Ex6) MMSE Estimation for Gaussian

- The temperature in a class room is known to follow Gaussian distribution $T \sim \mathcal{N}(10{,}2)\ C^o$. A thermometer is known to measure the temperature with noise, and can be modeled as $T_m = T + v$ with Gaussian noise $v \sim \mathcal{N}(0{,}1)$. You measured the temperature now and the thermometer says $T_m = 12$. What is the best estimate $\hat{T}$ of the actual temperature (in the sense of MMSE)? As always, we can assume the sensor noise v is independent from all other variables.

$$T_m = T + v$$

According to MMSE: $\hat{T} = E[T \mid T_m]$

$$= \mu_T + X_{TT_m} X_{T_mT_m}^{-1} (T_m - \mu_{T_m})$$

$$T \sim N(10,2), \quad v \sim N(0,1)$$

$$\mu_T = 10$$

$$\mu_{T_m} = E[T_m] = E[T + v] = E[T] + \cancel{E[v]} = \mu_T = 10$$

$$X_{TT_m} = E[(T - \mu_T)(T_m - \mu_{T_m})]$$

$$= E[(T - \mu_T)(T + v - \mu_T)]$$

$$= E[(T - \mu_T)^2] + \underbrace{\cancel{E[(T - \mu_T)v]}}_{\text{independent}} = X_{TT} = 2$$

$$X_{T_mT_m} = E[(T_m - \mu_{T_m})^2]$$

$$= E[(T + v - \mu_T)^2]$$

$$= E[(T - \mu_T)^2 + 2v(T - \mu_T) + v^2]$$

$$= E[(T - \mu_T)^2] + 2E[v(T - \mu_T)] + E[v^2]$$

$$= 2 + 0 + 1 = 3$$

$$\therefore \hat{T} = \mu_T + X_{TT_m} X_{T_mT_m}^{-1} (T_m - \mu_{T_m})$$

$$= 10 + 2 \cdot \tfrac{1}{3}(12 - 10)$$

$$= \boxed{11.33^\circ C}$$