Ex4) Estimating Noisy Binary Signal

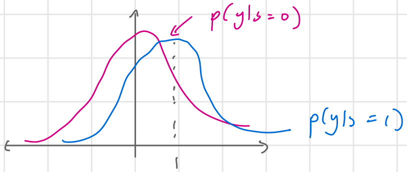
- A binary signal s (either 0 or 1) is transmitted randomly with $P(s=0)=p$ and $P(s=1)=1-p$ for some $p \in (0,1)$. The signal received at the receiver is analog and corrupted by Gaussian noise, i.e. $y = s + n, n \sim \mathcal{N}(0,1)$.
- Assume $p = 0.3$. Given a measurement $y = 0.1$,
  - what is the probability that $s=0$ (or $s=1$)?
  - which one is a better estimate of s between $\hat{s}=0$ and $\hat{s}=1$?

Given the observation/evidence y, the probability of $s=0$ is $p(s=0 \mid y)$.

$$p(s=0 \mid y) = \frac{p(y \mid s=0)\, p(s=0)}{p(y)}$$

$$= \frac{p(y \mid s=0)\, p(s=0)}{p(y \mid s=0)\, p(s=0) + p(y \mid s=1)\, p(s=1)}$$

Given s, the random variable y is a Gaussian with its mean at s: $E[y \mid s] = s$



$$= \frac{\frac{1}{\sqrt{2\pi}\,(1)} \exp\left(-\frac{(y-0)^2}{2}\right) \cdot p}{\frac{1}{\sqrt{2\pi}\,(1)} \exp\left(-\frac{(y-0)^2}{2}\right) \cdot p + \frac{1}{\sqrt{2\pi}\cdot 1} \exp\left(-\frac{(y-1)^2}{2}\right) \cdot (1-p)}$$

$$= \frac{p}{p + (1-p)\exp\left(y-\frac{1}{2}\right)}$$

$$= \frac{0.3}{0.3 + (0.7)\,e^{-0.4}} = 0.39$$

$p(s=1 \mid y=) = 1 - 0.39 = 0.61 \implies \boxed{\hat{s} = 1}$