Ex2) Bivariate Gaussian RV

- Assume a normal (Gaussian) random variable $x \sim \mathcal{N}(0, 2)$ Another random variable is generated as $y = 0.5x + v$ where $v \sim \mathcal{N}(0, 1)$ and v and x are independent. Find the covariance matrix for the (bivariate) random vector $[x \quad y]^T$, i.e. $\begin{bmatrix} X_{xx} & X_{xy} \\ X_{yx} & X_{yy} \end{bmatrix} \in \mathbb{R}^{2\times2}$

$E[x] = 0, \quad E[y] = E[0.5x + v] = 0.5E[x] + E[v] = 0$

$X_{xx} = Var(x) = E[(x-\mu_x)^2] = \underset{\mu_x = 0}{E[x^2]} = \sigma_x^2 = 2$

$X_{xy} = Cov(x,y) = E[(x-\mu_x)(y-\mu_y)] = E[x(0.5x + v - \overset{\mu_y = 0}{\mu_y})]$

$= E[x(0.5x + v)]$

$= E[0.5x^2 + xv]$

$= 0.5\sigma_x^2 + \cancel{E[xv]}$ $\begin{array}{l} = E[x]E[v] \text{ because they are independent} \\ = 0 \end{array}$

$= 1$

$X_{yy} = E[(y-\mu_y)^2] = E[(0.5x+v)^2]$

$= E[0.25x^2 + xv + v^2]$

$= 0.25\sigma_x^2 + E[xv] + \sigma_v^2$

$= 0.5 + 0 + 1 = 1.5$

$$\boxed{\Sigma = \begin{bmatrix} 2 & 1 \\ 1 & 1.5 \end{bmatrix}}$$